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Scalar solitons on the fuzzy sphere

Peter Austing, Thordur Jonsson and Larus Thorlacius

Science Institute, University of Iceland
Dunhaga 3, 107 Reykjavik, Iceland
E-mail: austing@raunvis.hi.is, thjons@raunvis.hi.is, lth@raunvis.hi.is

Abstract: We study scalar solitons on the fuzzy sphere at arbitrary radius and noncommutativity. We prove that no solitons exist if the radius is below a certain value. Solitons do exist for radii above a critical value which depends on the noncommutativity parameter. We construct a family of soliton solutions which are stable and which converge to solitons on the Moyal plane in an appropriate limit. These solutions are rotationally symmetric about an axis and have no allowed deformations. Solitons that describe multiple lumps on the fuzzy sphere can also be constructed but they are not stable.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry.

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1. Introduction

It is an interesting fact that the classical equations of motion of noncommutative field theory have a richer set of solutions than their commutative counterparts. Noncommutative solitons include deformations of conventional solitons but also new objects that are absent in the commutative theory. This applies to both gauge theories and scalar field theories and can be traced to the existence of a new length scale associated with the noncommutativity. A striking example is provided by scalar theory on the Moyal plane [1], where the corresponding commutative theory can have no solitons by Derrick's theorem [2]. Several authors have found explicit soliton solutions in gauge theory with and without matter fields, see, e.g., [3]–[6] and noncommutative deformations of commutative solitons have been studied in [7, 8, 9]. For reviews of noncommutative quantum field theory we refer to [10, 11].

The scalar solitons on the Moyal plane were first observed in the limit of infinite noncommutativity where the kinetic term in the action can be neglected and solutions correspond to multiples of projectors [1]. This line of reasoning was generalised to scalar fields on arbitrary Kähler manifolds including the sphere in [12] and subsequently multi-solitons on the fuzzy sphere in the infinite noncommutativity limit were considered in [13]. In [14, 15] it was shown that there exist scalar solitons on the Moyal plane at large but finite values of the noncommutativity parameter and that no soliton solutions exist depending smoothly on small values of the noncommutativity parameter as expected from Derrick's theorem. All the solitons at finite noncommutativity constructed so far have the property

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that they are rotationally invariant about some point. It is not known whether there exist stable soliton solutions in the plane describing separated lumps, but there are good reasons to believe that no such solutions exist and that all soliton solutions in the plane are rotationally invariant about some point. First of all, separated lumps attract each other according to perturbative calculations at large noncommutativity parameter [16, 17]. Furthermore, it has recently been proven that there cannot exist a family of static solutions on the Moyal plane at finite noncommutativity that interpolates smoothly between the solution describing two overlapping solitons and a solution with two infinitely separated solitons [18]. Finally, the stability results for scalar solitons on the fuzzy sphere obtained below, combined with a scaling limit that yields the Moyal plane from the fuzzy sphere (see e.g. [19]), provide further evidence for the non-existence of stable multi-lump solutions. Further results for noncommutative theory in the plane can be found in [20]–[25], and on the fuzzy sphere [26]–[33].

In the present paper, we study the equations of motion for a scalar field on the fuzzy sphere in detail, paying particular attention to issues that arise at finite noncommutativity. Part of our analysis is parallel to that of [14, 15] which dealt with existence and stability of scalar solitons on the Moyal plane. The algebra underlying the fuzzy sphere is finite dimensional, and this simplifies the mathematics, especially the stability analysis. It enables us to establish some uniqueness results by ruling out deformations of the field profile of a stable soliton. We are also able to prove a stronger nonexistence result at small noncommutativity than is available on the Moyal plane.

The fuzzy sphere is characterised by two parameters: R which can be interpreted as the radius of the sphere, and j which labels the associated matrix algebra. Together these parameters determine the strength of noncommutativity. A scalar field theory on the fuzzy sphere is further characterised by a scalar field potential $V(\phi)$ which we take to be a double well potential with a global minimum at the origin. Our main results are then the following:

- There are no solitons for small values of the radius.
- As the radius is increased a large number of soliton solutions come into existence, most of which are unstable.
- A necessary condition for a soliton, that is rotationally invariant around the north-south axis, to be stable is that its eigenvalues form a decreasing sequence (in the standard $su(2)$ basis). At sufficiently large radius, another condition is that the eigenvalues lie close to the minima of the potential, and these two conditions are also sufficient.
- The only small deformations of a stable soliton which lead to new soliton solutions are rotations.
- There is a family of stable, rotationally invariant solitons on the fuzzy sphere which converge to stable solitons in the Moyal plane as both R and j tend to infinity keeping R^2/j fixed and sufficiently large.

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The limit of infinite noncommutativity is by no means smooth, and the set of classical solutions is drastically reduced at finite as compared to infinite noncommutativity. The difference is particularly striking when one considers multisolitons. In fact, the very notion of a multisoliton solution is somewhat problematic at finite noncommutativity since there is no precise definition of soliton number in this case. In the limit of infinite noncommutativity, the soliton number of a stable classical solution can be taken to be the rank of the corresponding projector, but solutions at finite noncommutativity are no longer given by projectors and this definition fails. A physicist might wish to define a multisoliton as a configuration with two or more localised lumps in the scalar field that are separated on the fuzzy sphere, but this is not a precise mathematical definition. Our results strongly suggest that there are in fact no stable solutions with separated lumps, and that all stable solutions are rotationally symmetric. There is, in other words, no moduli space of scalar multisolitons on the fuzzy sphere at finite noncommutativity. We do, however, give evidence for multi-soliton solutions on the fuzzy sphere describing separated lumps that are evenly distributed along a great circle. These configurations are unstable due to the mutual attraction between lumps. In the special case of two identical lumps at the north and south poles, we give rigorous arguments for the existence of the unstable solutions.

By taking an appropriate limit $j, R \to \infty$, one can blow up the region close to the south pole into the Moyal plane. Assuming that every solution in the Moyal plane descends from solutions on the fuzzy sphere in the limit, one would expect a solution with separated lumps on the Moyal plane to descend from an infinitesimal deformation that breaks the rotation symmetry of a solution on the fuzzy sphere. The fact that no such deformations exist strongly suggests that there are no stable multi-lump solutions at finite theta on the Moyal plane, in agreement with the results of [18] and expectations from perturbative calculations in the limit of infinite non-commutativity [16, 17].

The unstable great circle solitons are lost in the Moyal plane limit since any lump not sitting at the south pole will be moved to infinity. By symmetry we would, however, expect the existence of an unstable solution consisting of an infinite chain of lumps finitely separated in a straight line on the Moyal plane. Such a solution would be periodic in one direction and have infinite energy. It could presumably be further generalized to a regular two-dimensional array of lumps.

The paper is organized as follows. In the following section we give a brief introduction to the fuzzy sphere and establish our notation. In section 3 we prove the nonexistence of static soliton solutions at small radius. In section 4 we study unstable solitons which bifurcate from the trivial soliton as the radius is increased. We construct some rotationally symmetric solitons on the fuzzy sphere in section 5, show that they are stable in section 6, and that they converge to solitons on the Moyal plane in an appropriate limit in section 8. As a byproduct of the stability analysis in section 6, we show that the only possible deformations of these stable solitons correspond to rotations that do not affect the shape of the field profile. In section 7 we establish the existence of (unstable) solutions that correspond to separate lumps at the north and south poles and give arguments for their generalization to lumps evenly distributed along a great circle.

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2. The fuzzy sphere

One of the simplest interesting noncommutative spaces is the fuzzy sphere [34] which can be regarded as an approximation to the true sphere obtained by cutting off the function algebra on the sphere by requiring the total angular momentum quantum number to be smaller than a prescribed value j. This requires of course that the ordinary product of functions be modified to a noncommutative product. If x_1, x_2 and x_3 are cartesian coordinates in $\mathbb{R}^3$ and the sphere is defined as

$$x_1^2 + x_2^2 + x_3^2 = R^2 \,, \tag{2.1}$$

then the commutation relations between the x_i's become

$$[x_j, x_k] = i\sigma\epsilon_{jk\ell}x_\ell \,. \tag{2.2}$$

Here σ is a positive real parameter. For our purposes it is convenient to regard the fuzzy sphere as the algebra M_j of $(2j+1) \times (2j+1)$ matrices, where j is a half integer. These matrices act on a $(2j+1)$-dimensional vector space which we denote by W_j. In M_j we have the spin matrices ($su(2)$ generators) for spin j which satisfy the usual commutation relation

$$[J_j, J_k] = i\epsilon_{jk\ell}J_\ell \,, \tag{2.3}$$

and $J_iJ_i = j(j+1)$. These matrices can be seen to generate the full matrix algebra M_j. With the above conventions we see that the x_i are proportional to the angular momentum generators, $x_i = \sigma J_i$, and the noncommutativity parameter σ is governed by the spin j and and the radius of the sphere through

$$R^2 = j(j+1)\sigma^2 \,. \tag{2.4}$$

The commutative sphere of radius R is recovered in the limit $j \to \infty$, keeping R fixed. Alternatively, one can take a scaling limit $j, R \to \infty$, keeping the ratio R^2/j fixed, and consider a small neighborhood around the south pole. The fuzzy sphere commutation relations (2.2) can then be shown to reduce to those of the Moyal plane [19].

Scalar field theory on the fuzzy sphere [35] is defined by the action functional

$$S = \frac{4\pi}{2j+1}\mathrm{Tr}\,\big([J_i, \phi]\,[\phi, J_i] + R^2V[\phi]\big) \tag{2.5}$$

where V is the potential and ϕ, the field, is an arbitrary hermitian $(2j+1) \times (2j+1)$ matrix. The action is invariant under rotations $\phi \to \mathcal{R}\phi\mathcal{R}^{-1}$ where $\mathcal{R} = \exp(i\theta\hat{n}_iJ_i)$. The variational equation of S is

$$2\,[J_i, [J_i, \phi]] + R^2V'(\phi) = 0 \,. \tag{2.6}$$

It follows immediately that any solution to this equation satisfies the condition

$$\mathrm{Tr}\,V'(\phi) = 0 \,. \tag{2.7}$$

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Introducing the raising and lowering operators $J_\pm = J_1 \pm iJ_2$, the variational equation can be written

$$[J_-, [J_+, \phi]] + [J_3, \phi] + [J_3, [J_3, \phi]] + \frac{R^2}{2} V'(\phi) = 0. \tag{2.8}$$

Let $|m\rangle$ denote the standard basis for W_j consisting of the eigenvectors of J_3, $J_3|m\rangle = m|m\rangle$, $m = -j, -j+1, \dots, j$. If a solution ϕ to eq. (2.8) is diagonal with respect to this basis, its matrix elements $\phi_m = \langle m|\phi|m\rangle$ satisfy the equation

$$\alpha_m \left(\phi_{m+1} - \phi_m\right) - \alpha_{m-1}\left(\phi_m - \phi_{m-1}\right) = \frac{R^2}{2} V'\left(\phi_m\right), \tag{2.9}$$

for $m = -j, -j+1, \dots, j$, where $\alpha_m = j(j+1) - m(m+1)$. It is convenient to formally introduce ϕ_{j+1} and ϕ_{-j-1} with the convention that $\phi_{j+1} = \phi_j$ and $\phi_{-j-1} = \phi_{-j}$. Summing eq. (2.9) over m the left hand side telescopes and we obtain the first order difference equation

$$\phi_{m+1} - \phi_m = \frac{R^2}{2\alpha_m} \sum_{i=-j}^{m} V'(\phi_i), \qquad m = -j, \dots, j-1 \tag{2.10}$$

with the constraint

$$\sum_{i=-j}^{j} V'(\phi_i) = 0. \tag{2.11}$$

Diagonal matrices commute with J_3 and diagonal solutions of the scalar field equations correspond precisely to solitons which are invariant under rotations about the z-axis.

We shall restrict ourselves to studying potentials $V(x)$ which are twice continuously differentiable, nonnegative, and having a double zero at $x = 0$ and $V(x) > 0$ for $x \neq 0$. We shall assume $V(x)$ has a local minimum (false vacuum) at $s > 0$ in addition to the global minimum (true vacuum) at $x = 0$, that is $V'(x)$ has zeros at $x = 0$, $x = r$ and $x = s$ with $0 < r < s$, and no others (see figure 1).

We first note that any solution ϕ to the equations of motion (2.6) has its spectrum in the interval $[0, s]$. A proof of the equivalent result for scalar field theory on the Moyal plane

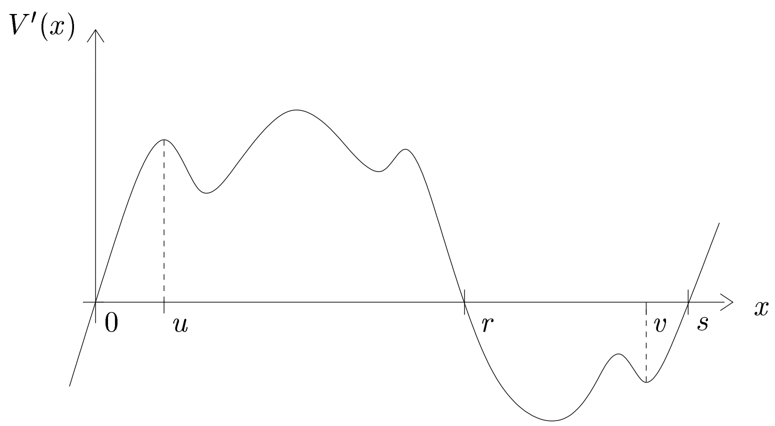

Figure 1: An example of the derivative of a double-well potential satisfying our assumptions. The points u and v are the critical points of V' closest to 0 and s, respectively.

is given by the authors of [15]. This proof carries over immediately to the fuzzy sphere so we do not reproduce it here.

If ϕ is any operator on W_j we denote its matrix elements $\langle m|\phi|n\rangle$ by $\phi_{m,n}$. Before proceeding, we note that, in addition to rotational symmetry, the soliton equation (2.6) is invariant under the transformation

$$\phi_{m,n} \to \phi_{-m,-n} \tag{2.12}$$

which corresponds to reflection in the equator. Then, for example, if ϕ is a diagonal soliton, so is ψ, where $\psi_m = \phi_{-m}$.

3. Nonexistence at small R

In this section we show that, at fixed j, there are no solitons when the radius R is sufficiently small. For convenience we define $\mu = R^2/2$. We define an inner product and a corresponding norm on hermitian matrices by

$$(\phi, \psi) = \mathrm{Tr}\,(\phi\psi)\,, \qquad ||\phi||^2 = \mathrm{Tr}\,\phi^2\,. \tag{3.1}$$

Let ϕ be a solution to the soliton equation (2.6) and let I denote the unit matrix. We begin by showing that

$$||\phi - rI|| = \mathcal{O}(\mu)\,. \tag{3.2}$$

The notation $f = \mathcal{O}(\psi^n)$ will mean $||f|| = \mathcal{O}\left(||\psi||^n\right)$.

In order to prove (3.2) we note that any hermitian matrix ϕ can be split into two pieces

$$\phi = \phi_{||} + \phi_{\perp} \tag{3.3}$$

with $\phi_{||}$ proportional to the identity matrix, and $(\phi_{||}, \phi_{\perp}) = 0$ (so that $\mathrm{Tr}\,\phi_{\perp} = 0$). The laplacian operator (total angular momentum operator) in (2.6) has eigenvalues $k(k+1)$ with $k = 0, 1, \ldots, 2j$. The unique eigenstate with $k = 0$ is the identity matrix. Substituting ϕ into (2.6) gives

$$[J_i, [J_i, \phi_{\perp}]] + \mu V'(\phi) = 0 \tag{3.4}$$

so that

$$\mu\big|\big|V'(\phi)\big|\big| \geq 2||\phi_{\perp}||\,. \tag{3.5}$$

Since any solution ϕ has its eigenvalues in the interval $[0, s]$, we find that

$$||\phi_{\perp}|| \leq c\mu, \tag{3.6}$$

where c is a constant. Next, we look at the trace condition

$$\mathrm{Tr}\,V'\left(\phi_{||} + \phi_{\perp}\right) = 0\,. \tag{3.7}$$

Then

$$\mathrm{Tr}\,V'\left(\phi_{||}\right) + \mathcal{O}(\mu) = 0 \tag{3.8}$$

and we conclude that for small values of μ

$$\phi_{||} = x_0 I + \mathcal{O}(\mu)\,, \tag{3.9}$$

where x_0 is a zero of V', and so

$$\phi = x_0 I + \mathcal{O}(\mu)\,. \tag{3.10}$$

Next, we diagonalise ϕ and discover eigenvalues $\phi_m = x_0 + \mathcal{O}(\mu)$. Since every eigenvalue lies in the range $[0, s]$, the trace condition implies that there must be some eigenvalues smaller than r and some larger than r. So by taking μ small enough, we must have $x_0 = r$ and this proves equation (3.2).

If we now write

$$\phi = rI + \chi\,, \qquad \chi = \chi_{||} + \chi_{\perp}\,, \tag{3.11}$$

using the splitting (3.3), and substitute into the soliton equation (2.6), we obtain similarly

$$\mu\big|\big|V'(\phi)\big|\big| \geq 2||\chi_{\perp}||\,. \tag{3.12}$$

On the other hand we have

$$\begin{aligned} \big|\big|V'(\phi)\big|\big|^2 &= \mathrm{Tr}\, V'(rI+\chi)^2 \\ &= V''(r)^2||\chi||^2 + \mathcal{O}\left(\chi^3\right) \\ &= V''(r)^2\left(||\chi_{\perp}||^2 + \big|\big|\chi_{||}\big|\big|^2\right) + \mathcal{O}\left(\chi^3\right)\,. \end{aligned} \tag{3.13}$$

The trace condition gives

$$\mathrm{Tr}\, V'(\phi) = \mathrm{Tr}\,\left(\chi_{\perp} + \chi_{||}\right) V''(r) + \mathcal{O}\left(\chi^2\right) \tag{3.14}$$

and since $\mathrm{Tr}\,\chi_{\perp} = 0$,

$$\big|\big|\chi_{||}\big|\big|^2 = \mathcal{O}\left(\chi^4\right) \tag{3.15}$$

so

$$\big|\big|V'(\phi)\big|\big|^2 = V''(r)^2||\chi_{\perp}||^2 + \mathcal{O}\left(\chi^3\right)\,. \tag{3.16}$$

Comparing this with (3.12) we find

$$\mu^2\left(V''(r)^2||\chi_{\perp}||^2 + \mathcal{O}\left(\chi^3\right)\right) \geq 4||\chi_{\perp}||^2\,. \tag{3.17}$$

Noting that $||\chi_{\perp}||^2 = ||\chi||^2 - \big|\big|\chi_{||}\big|\big|^2 = ||\chi||^2\left(1 + \mathcal{O}\left(\chi^2\right)\right)$ we obtain finally

$$\mu^2\left(V''(r)^2 + \mathcal{O}\left(\chi^3\right)\right) \geq 4\,. \tag{3.18}$$

If μ is small enough, this is impossible.

Recall that at fixed j the radius R determines the non-commutativity parameter σ through (2.4). We have thus established the non-existence of scalar solitons at sufficiently weak non-commutativity on the sphere. This is certainly in line with previous results on the Moyal plane [14] but this was not guaranteed *a priori* given that Derrick's theorem only holds on the commutative plane but not on the sphere.

4. Existence close to critical values of R

In this section we establish the existence of solutions to the diagonal soliton equations (2.10) and (2.11) at certain critical values of the radius R. When we perform our stability analysis in section 6, these solutions will all turn out to be unstable. They do, however, provide insight into the structure of the field equations and we include them before moving on to the construction in section 5, which yields stable solutions for sufficiently large R.

We think of ϕ_{-j} as the first entry in the matrix and then use the first order difference equation (2.10) to find each ϕ_m for $m > -j$ as a function of $\phi_{-j} \equiv x$. In order to discover whether a particular value of x gives a soliton solution, we only need to check the constraint (2.11). This is equivalent to looking for zeros of the function

$$g_\mu(x) = \sum_{i=-j}^{j} V'\left[\phi_i(x)\right] \tag{4.1}$$

where $\mu = R^2/2$. We know that there are (trivial) solutions to the soliton equations at $x = 0, r, s$ so $g_\mu(x)$ must have zeros at these points. We also know that when μ is small, $g_\mu(x)$ has no other zeros. Also, if $x < 0$, the $\phi_m(x)$ get locked into the negative region of V' and so $g_\mu(x) < 0$. Similarly, when $x > s$, $g_\mu(x) > 0$.

Assuming that solitons do exist at some values of μ, we can imagine increasing μ to the critical point μ_c at which they come into existence. Imagine plotting a graph of $g_\mu(x)$ against x. As we move through the critical value of μ, $g_\mu(x)$ will brush and then cut through the x axis. This will happen at μ_c, x_c satisfying

$$g_{\mu_c}(x_c) = g'_{\mu_c}(x_c) = 0\,, \tag{4.2}$$

and by the continuity of $g_\mu(x)$, we will generically pick up a pair of new solitons.

In order to find the critical points, one in general needs to solve a nonlinear equation. There is, however, a series of critical points with $x_c = r$ which we can determine analytically.

Setting $\phi_{-j} = x = r$, we solve the soliton equations and find $\phi_m = r$ for all m, so that $g_\mu(r) = 0$. Differentiating eq. $\tilde{(}4.1)$ with respect to x and putting $x = r$ leads to the constraint

$$g'_\mu(r) = V''(r) \sum_{i=-j}^{j} \phi'_i(r) = 0\,. \tag{4.3}$$

We can differentiate (2.9) to obtain a difference equation for the $\phi'_i(r)$, with initial value $\phi'_{-j}(r) = 1$

$$\alpha_m\left(\phi'_{m+1}(r) - \phi'_m(r)\right) - \alpha_{m-1}\left(\phi'_m(r) - \phi'_{m-1}(r)\right) = \mu V''(r)\phi'_m(r)\,. \tag{4.4}$$

Let us define a diagonal matrix u with $u_{ii} = \phi'_i(r)$ (note that, although we have not explicitly written it, the matrix u is a function of μ, inherited from the definition of the ϕ_i). As long as $\mu \neq 0$, we can combine (4.3) with (4.4) to obtain an equivalent matrix equation

$$[J_i, [J_i, u]] + \mu V''(r) u = 0\,. \tag{4.5}$$

The laplacian has eigenvalues $k(k+1)$, with $k = 0, 1, 2, \ldots, 2j$, and there is precisely one eigenstate represented by a diagonal matrix e_k for each eigenvalue. Therefore the critical values of μ at $x_c = r$ are

$$\mu_c = \frac{-k(k+1)}{V''(r)}, \qquad k = 1, 2, \ldots, 2j\,, \tag{4.6}$$

and at the critical point we have

$$u(\mu_c) = a e_k, \qquad a \neq 0\,. \tag{4.7}$$

We note that the number of critical points increases with j but the lowest critical value of μ (and therefore of R) is independent of j.

The picture that we have is the following. As we increase μ towards a critical point, the function $g(x)$ flattens at $x = r$, and as we move through the critical point, we expect g to start to cut through the x-axis on either side of $x = r$ so that we gain two new solitons. Alternatively, it is possible that this process will precisely correspond with another pair of solitons moving in towards the critical point leading to a cancellation so that the net effect is the loss of a pair of solitons. In any case, when $g'(r)$ changes sign, we must either gain or lose two soliton solutions since $g(x)$ is continuous.

There is an additional possibility which we cannot yet discount. Even though $g'(r) = 0$ at the critical value of μ, it is possible that it may not change sign as we move through the critical value. This can only be if

$$\frac{\partial}{\partial \mu}\, g'_\mu(r)|_{\mu=\mu_c} = 0\,. \tag{4.8}$$

If we assume (4.8) is true, then by a very similar argument to the previous, we obtain the matrix equation

$$[J_i, [J_i, \dot{u}(\mu_c)]] + V''(r) u\,(\mu_c) + \mu_c V''(r)\dot{u}(\mu_c) = 0 \tag{4.9}$$

where $\dot{u}$ denotes u differentiated with respect to μ. As before, we can expand $u(\mu)$ in terms of the eigenstates of the laplacian $u(\mu) = \sum_{l=0}^{2j} u_l(\mu) e_l$. We are studying a critical point satisfying $\mu_c V''(r) = -k(k+1)$ for some fixed k, so let us pick out the coefficient of e_k:

$$0 = k(k+1)\dot{u}_k(\mu_c) + V''(r) u_k(\mu_c) + \mu_c V''(r)\dot{u}_k(\mu_c) = V''(r) u_k(\mu_c)\,. \tag{4.10}$$

But by (4.7), $u(\mu_c) = u_k(\mu_c) e_k \neq 0$, so this is a contradiction.

The conclusion is that whenever μ moves through one of the critical points (4.6), we either gain or lose two solitons. As a corollary, there is a range of μ close to each of the critical points for which solitons exist.

To illustrate, figure 2 shows an example plot of the zeros of $g_\mu(x)$ for $j = \frac{3}{2}$ and a potential with $V'(u) = u(u - 1.2)(u - 2.0)$. We see that there are many critical points in addition to those we have calculated. In most cases, soliton solutions are created as we increase μ, but they are also occasionally lost. In some cases, the critical points occur as bifurcations of existing solitons, but in most cases a pair of solitons appears (or disappears) where there was none before. Nevertheless, we call this the bifurcation diagram. We have observed that if one increases j, the complexity of the bifurcation diagram increases, and the number of solitons at sufficiently large μ increases.

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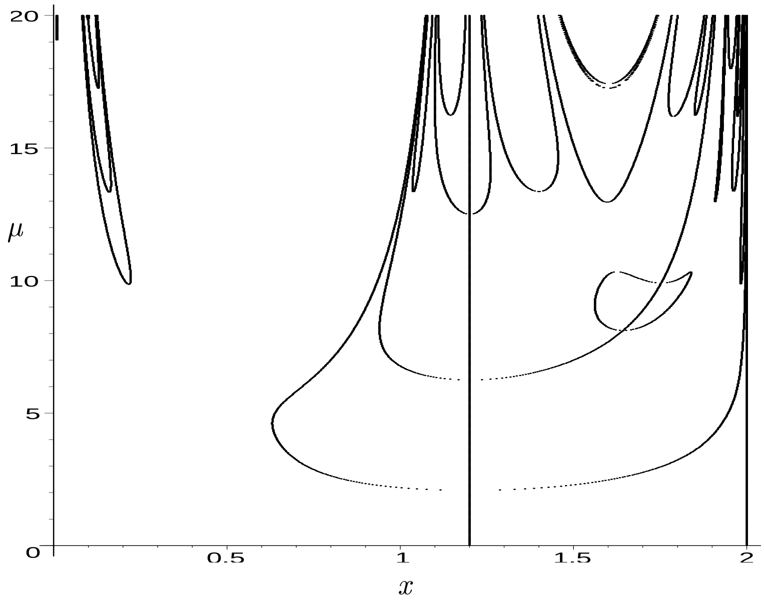

Figure 2: Diagram showing the zeros of $g_\mu(x)$ for $j = 3/2$.

5. Diagonal solitons

We now show, for any given value of j, that the diagonal soliton equations (2.10) and (2.11) have a solution provided R is large enough. In section 6 we will further show that this construction leads to stable solutions. As j increases, we will need to increase R^2 linearly, so let us write $R^2/2 = \theta j$ at the outset. When we come to consider the planar limit, it will also be convenient to rewrite the matrix elements of ϕ

$$\phi = \sum_{m=-j}^{j} \phi_m |m\rangle\langle m| = \sum_{q=0}^{2j} \psi_q^{(j)} |-j+q\rangle\langle -j+q| \tag{5.1}$$

so that $\psi_q^{(j)} = \phi_{-j+q}$. Then the equations of motion (2.10), (2.11) become

$$\psi_{q+1}^{(j)} - \psi_q^{(j)} = \frac{\theta}{2(q+1)(1-q/2j)} \sum_{k=0}^{q} V'\left(\psi_k^{(j)}\right), \qquad q = 0, \ldots, 2j-1, \tag{5.2}$$

$$\sum_{k=0}^{2j} V'(\psi_k^{(j)}) = 0\,. \tag{5.3}$$

We have included an index j in $\psi_q^{(j)}$ to remind ourselves that we are working with the fuzzy sphere, M_j, which is the j-th representation of $su(2)$.

We will see that solutions do exist for sufficiently large θ by adapting the proof given in [14] for the Moyal plane.

Theorem 1. *There is a constant, θ_c, independent of j such that when $\theta > \theta_c$ equations (5.2), (5.3) have a non-trivial solution.*

To see this, suppose V' has its first local maximum in the domain $[0,s]$ at u and its final local minimum at v (see figure 1). We begin by choosing

$$\theta > \frac{-2v}{V'(v)}\,. \tag{5.4}$$

Then setting $q=0$ in (5.2), we have

$$\psi_1^{(j)} = \psi_0^{(j)} + \frac{\theta}{2} V'\left(\psi_0^{(j)}\right), \tag{5.5}$$

so there is a unique x with $v < x < s$ such that if $\psi_0^{(j)} = x$ then $\psi_1^{(j)} = 0$, and this x is independent of j. As θ increases, x moves towards s independently of j. Then we can choose θ sufficiently large that

$$\left|V'(x)\right| < V'(u). \tag{5.6}$$

This is the final constraint on θ, so our critical value θ_c will indeed be independent of j.

Now if we increase $\psi_0^{(j)}$ from x towards s, $\psi_1^{(j)}$ increases to s. Therefore, we can choose y such that $x < y < s$ and when $\psi_0^{(j)} = y$ then $\psi_1^{(j)} = u$.

Using (5.2), each $\psi_0^{(j)}$ defines a sequence $\psi_k^{(j)}$. We next define the set

$$A^{(j)} = \left\{ \psi_0^{(j)} \in [x,y) : \exists q, 0 \le q \le 2j, \sum_{k=0}^{q} V'\left(\psi_k^{(j)}\right) > 0 \right\}. \tag{5.7}$$

We see from (5.2) that if $\psi_0^{(j)}$ leads to a sequence $\psi_k^{(j)}$ which is not monotonically decreasing, then $\psi_0^{(j)}$ is in $A^{(j)}$. In addition there may be a monotonically decreasing sequence for which $\psi_0^{(j)}$ is in $A^{(j)}$, if $\sum_{k=0}^{2j} V'\left(\psi_k^{(j)}\right) > 0$.

We note that $x \notin A^{(j)}$, since the sequence which starts at x jumps first to 0, and then gets locked into the negative region so continues to decrease. On the other hand, using (5.6), we see that $y - \epsilon \in A^{(j)}$ as long as ϵ is sufficiently small. So $A^{(j)}$ is not empty. Since V' is continuous, we also note that $A^{(j)}$ is open.

Following [14], we choose

$$\psi_0^{(j)} = \inf A^{(j)}\,. \tag{5.8}$$

Then $\psi_0^{(j)} \notin A^{(j)}$ since $A^{(j)}$ is open, and so

$$\sum_{k=0}^{q} V'\left(\psi_k^{(j)}\right) \le 0\,, \qquad q = 0, \ldots, 2j\,. \tag{5.9}$$

But since $\psi_0^{(j)}$ is the infimum of $A^{(j)}$, there must by continuity be at least one q such that

$$\sum_{k=0}^{q} V'\left(\psi_k^{(j)}\right) = 0 \tag{5.10}$$

and we choose $\tilde{q}$ to be the smallest such q.

Let us suppose $\tilde{q} < 2j$. Then since $\sum_{k=0}^{\tilde{q}-1} V'\left(\psi_k^{(j)}\right) < 0$, we have certainly $V'\left(\psi_{\tilde{q}}^{(j)}\right) > 0$. Using (5.2), $\psi_{\tilde{q}+1}^{(j)} = \psi_{\tilde{q}}^{(j)}$ and therefore $\sum_{k=0}^{\tilde{q}+1} V'\left(\psi_k^{(j)}\right) > 0$, which contradicts (5.9). We deduce that $\sum_{k=0}^{2j} V'\left(\psi_k^{(j)}\right) = 0$, and so we have completed the construction of a solution to the equations (5.2), (5.3) for sufficiently large θ. This solution has the property that the sequence $\psi_k^{(j)}$ is monotonically decreasing.

We can extend the proof given above (in the same manner as in [15]) to construct further soliton solutions. This time, we fix an integer N with $0 < N \leq 2j$ and require

$$\theta > \frac{-2vN}{V'(v)}. \tag{5.11}$$

Then there is a unique $x^{(j)}$ such that if $\psi_0^{(j)} = x^{(j)}$, then

$$\psi_0^{(j)} > \psi_1^{(j)} > \ldots > \psi_{N-1}^{(j)} \geq v \tag{5.12}$$

and $\psi_N^{(j)} = 0$.

Only in the case of $N = 1$ which we covered before is $x^{(j)}$ independent of j. However, we note that $x^{(j)} \geq x^{(j+1)}$, and define $\lim_{j\to\infty} x^{(j)} = x^{(\infty)} \in (v, s)$. If we take the limit of equation (5.2) as $j \to \infty$, we see that as we increase θ to infinity, $x^{(\infty)}$ increases to s. This means that we can make each $x^{(j)}$ arbitrarily close to s by increasing θ *independently of* j.

We next take θ sufficiently large that

$$N\left|V'\left(x^{(\infty)}\right)\right| < V'(u)\,. \tag{5.13}$$

This is the final constraint on θ and we note that it depends on N, but not j. We choose $y^{(j)} > x^{(j)}$ such that if $\psi_0^{(j)} = y^{(j)}$ then $\psi_{N+1}^{(j)} = u$. Replacing x and y with $x^{(j)}$ and $y^{(j)}$ in the definition (5.7) of $A^{(j)}$, the proof now proceeds as before.

We have constructed for each $N = 1, 2, \ldots, 2j$ a soliton with the property that the $\psi_q^{(j)}$ form a monotonically decreasing sequence. The first N eigenvalues lie in the interval (v, s) and the others lie in $(0, u)$. In particular, we note that each $\psi_q^{(j)}$ satisfies $V''\left(\psi_q^{(j)}\right) > 0$. In the large θ limit, the first N eigenvalues tend to s and the others to zero.

6. Stability

In this section, we investigate the stability of diagonal solitons. We will show that a necessary condition for stability is that the eigenvalues form a monotonic sequence. Next, we specialise to values of μ greater than some critical value μ_1 depending on the potential V and on j. Then another necessary condition for stability is that the eigenvalues lie in the regions close to the troughs of V in which $V'' > 0$. Together, these two conditions are also sufficient for stability, so that the solitons we constructed in section 5 are stable for sufficiently large μ. Finally, we show that the only deformations of a stable soliton which leave its energy invariant are rotations.

The tools for analysing stability have been set up and discussed in detail in [15] in the context of the Moyal plane (and see also [24] in which stability under radial perturbations was considered). The stability functional is defined

$$\Sigma(\omega) = \frac{1}{2}\frac{d^2}{d\epsilon^2}S(\phi + \epsilon\omega)\bigg|_{\epsilon=0} \tag{6.1}$$

where ϕ is a soliton solution. If Σ is nonnegative then the soliton ϕ is said to be stable. We will find that Σ always has zero eigenvalues because of the rotation symmetry of the action.

Substituting for the action, we can rewrite the stability functional as

$$\Sigma(\omega) = J(\omega) + \mu \frac{d^2}{d\epsilon^2} \mathrm{Tr}\, V(\phi + \epsilon\omega)\Big|_{\epsilon=0} \tag{6.2}$$

where $J(\omega) = \mathrm{Tr}\,[J_i, \omega][\omega, J_i]$ and $\mu = R^2/2$. Using standard non-degenerate perturbation theory (see [15]), this can be written

$$\Sigma(\omega) = J(\omega) + 2\mu \sum_{m<n} |\langle n|\omega|m\rangle|^2 \frac{V'(\phi_m) - V'(\phi_n)}{\phi_m - \phi_n} + \mu \sum_{n=-j}^{j} |\langle n|\omega|n\rangle|^2 V''(\phi_n) \tag{6.3}$$

as long as $\phi_m \neq \phi_n$ for all m, n. If there are degenerate eigenvalues $\phi_m = \phi_n$, we can use the fact that the expression (6.2) is continuous in ϕ, and obtain the correct formula by taking the limit of (6.3) as the eigenvalues approach each other. In this way, we see that we must make the substitution

$$\frac{V'(\phi_m) - V'(\phi_n)}{\phi_m - \phi_n} \to V''(\phi_m) \tag{6.4}$$

in equation (6.3).

Examining the equation of motion (2.6), we see that the eigenvalues must lie within $\mathcal{O}(\mu^{-1})$ of the zeros of V'. If any ϕ_m is close to r, then since $V''(r) < 0$, the last term in (6.3) can be made large and negative, so the soliton is unstable. So choosing μ sufficiently large, our first necessary condition for stability is that each ϕ_m lies in the region $[0, u]$ or $[v, s]$ (see figure 1). This condition simply says that the eigenvalues must all lie in the regions close to the troughs of V in which $V'' > 0$.

We can write

$$J(\omega) = \sum_{m,n} |\sqrt{\alpha_{n-1}}\langle n|\omega|m\rangle - \sqrt{\alpha_{m-1}}\langle n-1|\omega|m-1\rangle|^2 + (n-m)^2 |\langle n|\omega|m\rangle|^2 \tag{6.5}$$

and note that Σ is a quadratic form in the matrix elements of ω, and these matrix elements are only coupled along diagonals, which we label by an integer k. We dealt with the case $k = 0$ in the previous paragraph, and so the stability problem can be reduced to studying the reduced functional

$$\begin{aligned}\Sigma_k(\omega) = & \sum_{|n-m|=k} |\sqrt{\alpha_{n-1}}\langle n|\omega|m\rangle - \sqrt{\alpha_{m-1}}\langle n-1|\omega|m-1\rangle|^2 + \\ & + \sum_{|n-m|=k} (n-m)^2 |\langle n|\omega|m\rangle|^2 + \\ & + 2\mu \sum_{n-m=k} |\langle n|\omega|m\rangle|^2 \frac{V'(\phi_m) - V'(\phi_n)}{\phi_m - \phi_n}\end{aligned} \tag{6.6}$$

for each $k > 0$.

We fix k and define $x_n = \langle n+k|\omega|n\rangle$ for $n = -j, \ldots, j-k$, so that $x_n^* = \langle n|\omega|n+k\rangle$. Then we can write

$$\Sigma_k(\omega) = 2\sum_{m,n} q_{mn} x_m x_n^* \tag{6.7}$$

where the only non-zero elements of the quadratic form q are

$$q_{nn} = \alpha_n + \alpha_{n+k-1} + k(k-1) + \gamma_n \tag{6.8}$$

$$q_{n,n-1} = -\sqrt{\alpha_{n-1}\alpha_{n+k-1}} \tag{6.9}$$

and where

$$\gamma_n = \mu\frac{V'(\phi_{n+k}) - V'(\phi_n)}{\phi_{n+k} - \phi_n}\,, \qquad \phi_n \neq \phi_{n+k} \tag{6.10}$$

$$\gamma_n = \mu V''(\phi_n)\,, \qquad \phi_n = \phi_{n+k}\,. \tag{6.11}$$

To obtain (6.8), we used the identity

$$\frac{1}{2}(\alpha_{n+k-1} + \alpha_n + \alpha_{n-1} + \alpha_{n+k}) = \alpha_n + \alpha_{n+k-1} - k\,. \tag{6.12}$$

We will follow [15] to use elementary row and column operations to find a new diagonal quadratic form, C, which has the same numbers of positive, negative and zero eigenvalues as q. Specifically, we define the diagonal elements of C inductively by

$$C_{-j} = q_{-j,-j} \tag{6.13}$$

$$C_n = q_{n,n} - \frac{q_{n,n-1}^2}{C_{n-1}}\,, \qquad (n = -j+1, \ldots, j-k)\,. \tag{6.14}$$

We are now ready to prove another necessary condition for stability.

Theorem 2. *A necessary condition for stability is that the eigenvalues ϕ_n form a monotonic sequence.*

We note first that we do not need to assume μ is large.

We examine $\Sigma_k(\omega)$ with $k = 1$. We can use (2.9) to eliminate $V'(\phi_n)$ and $V'(\phi_{n+1})$ from γ_n, and find

$$q_{nn} = \alpha_{n+1}\frac{\phi_{n+2} - \phi_{n+1}}{\phi_{n+1} - \phi_n} + \alpha_{n-1}\frac{\phi_n - \phi_{n-1}}{\phi_{n+1} - \phi_n}\,, \tag{6.15}$$

as long as $\phi_n \neq \phi_{n+1}$. Let us assume first that there are no such pairs of consecutive degenerate eigenvalues. Then it is very easy to check by induction that

$$C_n = \alpha_{n+1}\frac{\phi_{n+2} - \phi_{n+1}}{\phi_{n+1} - \phi_n} \tag{6.16}$$

for all $n = -j, \ldots, j-1$. If the ϕ_n are not monotonic, at least one of the C_n must be negative.

For the degenerate case, assume that the first pair of consecutive degenerate eigenvalues is $\phi_p = \phi_{p+1}$. Then we obtain

$$C_n = \alpha_{n+1}\frac{\phi_{n+2} - \phi_{n+1}}{\phi_{n+1} - \phi_n}\,, \qquad n \leq p-1\,, \tag{6.17}$$

and in particular, $C_{p-1} = 0$ so that this inductive reduction fails at $n = p - 1$. However, we can use the $(p-1)$-th row and column to further eliminate and bring the matrix into block diagonal form, and then C_p and C_{p+1} are given by the eigenvalues of

$$\begin{pmatrix} 0 & -\sqrt{\alpha_{p-1}\alpha_{p-2}} \\ -\sqrt{\alpha_{p-1}\alpha_{p-2}} & 2\alpha_p + \mu V''(\phi_p) \end{pmatrix} \tag{6.18}$$

one of which is always negative. This completes the proof of theorem 2.

In equation (6.16), we note that C_{j-1} is zero (since $\alpha_j = 0$). This zero eigenvalue in the stability functional comes about because we can rotate the soliton, and obtain a new solution with the same energy. In general, if there is a family of soliton solutions $\phi(\nu)$ depending smoothly on ν, with $\phi(0) = \phi$, then necessarily there exists an $\omega \neq 0$ with $\Sigma(\omega) = 0$. To see this, take $\omega = d\phi/d\nu|_{\nu=0}$. We can always choose a parameterization such that $\omega \neq 0$. Then, expanding ϕ in terms of some basis for the matrices, $\phi = \phi^i T^i$, we have

$$0 = \frac{1}{2}\,\frac{d^2}{d\nu^2} S[\phi(\nu)]\bigg|_{\nu=0} = \frac{1}{2}\,\frac{\partial^2 S}{\partial\phi^i \partial\phi^j}\bigg|_{\nu=0} \omega^i \omega^j + \frac{\partial S}{\partial \phi^i}\,\frac{d^2\phi^i}{d\nu^2}\bigg|_{\nu=0} \tag{6.19}$$

The first term is the stability functional, whilst the second term is zero by the equation of motion, so we obtain $\Sigma(\omega) = 0$.

Since the rotations are a symmetry of the action, they will lead to zero eigenvalues of the stability functional. The rotations are a three parameter group, but one of these (rotation about the z-axis) leaves a diagonal soliton invariant. This means we would expect to find two zero eigenvalues in the stability functional corresponding to the rotations. The quadratic form q acts on a complex space, and so the zero eigenvalue C_{j-1} corresponds to two zero eigenvalues on a real space. Our task now is to show that this zero eigenvalue corresponds precisely to the rotations.

We do this by showing that any hermitian matrix $\mathcal{X} \in M_j$ can always be rotated into a canonical form.[1] Then we can restrict our space to matrices of this form, and check that the zero eigenvalues disappear. Specifically, given any hermitian matrix $\mathcal{X}_{pq}$, $(-j \leq p, q \leq j)$, we can always use a rotation to set $\mathcal{X}_{j,j-1} = \mathcal{X}_{j-1,j} = 0$, and we shall restrict to this rotation fixed space of matrices.

To see that this is possible, it is helpful to use the coherent state representation for the algebra. The formulae that we use can be found in [36]. A general rotation generated by J_1 and J_2 can be re-written in *normal form*

$$\exp(i\theta_1 J_1 + i\theta_2 J_2) = \exp(\zeta J_+)\exp(\eta J_3)\exp(-\bar{\zeta} J_-) \equiv D(\zeta) \tag{6.20}$$

where, defining $\theta = \theta_1 + i\theta_2$, we have $\zeta = i|\theta|^{-1}\tan(|\theta|/2)\,\theta$ and $\eta = \log(1 + |\zeta|^2)$. It can also be written in *antinormal form*

$$D(\zeta) = \exp\left(-\bar{\zeta} J_-\right)\exp(-\eta J_3)\exp(\zeta J_+)\,. \tag{6.21}$$

[1]We recall this notation from section 2. The fuzzy sphere is the algebra M_j of $(2j+1) \times (2j+1)$ complex matrices.

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The coherent state is defined

$$|z\rangle = D(z)|j,-j\rangle\,, \tag{6.22}$$

and is automatically normalised $\langle z|z\rangle = 1$.

The operators D satisfy

$$D(\zeta_1)D(\zeta_2) = D(\zeta_3)\exp(i\Phi J_3) \tag{6.23}$$

where ζ_3 and Φ are respectively complex and real numbers depending on ζ_1 and ζ_2. In particular, $D(\zeta)D(-\zeta) = 1$ as can be checked by multiplying the normal and antinormal forms.

Given a hermitian matrix $\mathcal{X} \in M_j$, we define the covariant symbol [37]

$$\mathcal{X}(z,\overline{z}) = \langle z|\mathcal{X}|z\rangle\,, \tag{6.24}$$

which, in terms of matrix elements in the standard basis, is

$$\mathcal{X}(z,\overline{z}) = \frac{1}{(1+|z|^2)^{2j}}\sum_{m,n=-j}^{j}\mathcal{X}_{mn}\overline{z}^{m+j}z^{n+j}\sqrt{\binom{2j}{m+j}\binom{2j}{n+j}}\,. \tag{6.25}$$

The covariant symbol is one way of mapping the fuzzy sphere algebra M_j to an algebra of functions on the sphere (with noncommutative $*$-product). Here the sphere is given in stereographic coordinates, and we note that if the matrix $\mathcal{X}$ is diagonal, $\mathcal{X}(z,\overline{z})$ depends only on $|z|$, so that it is indeed rotationally invariant about the north-south axis.

The function $\mathcal{X}(z,\overline{z})$ must have at least one stationary point, z_0, in $|z| < \infty$. One can quickly check using (6.23) that applying a rotation $\mathcal{X} \to \widetilde{\mathcal{X}} = D^\dagger(\zeta)\mathcal{X}D(\zeta)$ leads to a new covariant symbol $\widetilde{\mathcal{X}}(z,\overline{z}) = \mathcal{X}(z',\overline{z}')$ for some z' depending on z and ζ. In particular, we can apply the rotation $D(-z_0)$ to get a covariant symbol with a stationary point at $z = 0$. Then differentiating (6.25) at $z = 0$ gives

$$\widetilde{\mathcal{X}}_{j,j-1} = \widetilde{\mathcal{X}}_{j-1,j} = 0 \tag{6.26}$$

which is our desired rotation fixing condition.

We can now deal with the zero eigenvalue C_{j-1}. We restrict to the rotation fixed space of matrices $\mathcal{X}$, with $\mathcal{X}_{j,j-1} = \mathcal{X}_{j-1,j} = 0$. In this case, the above analysis works identically except that C_{j-1} no longer appears. This shows that the zero eigenvalue corresponds precisely to the rotations. In particular, when the eigenvalues form a monotonic sequence, Σ_1 is positive definite on the rotation fixed space.

We now go on and consider Σ_k for $k > 1$.

Theorem 3. *Suppose ϕ is a diagonal soliton, and the eigenvalues ϕ_m form a strictly monotonic sequence, and suppose μ is sufficiently large. Assume further that every eigenvalue ϕ_m has $V''(\phi_m) > 0$. Then the quadratic forms Σ_k are positive definite on the rotation fixed space.*

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We have already covered the cases $k = 0$ and $k = 1$, so we restrict now to $k > 1$. Since μ is large, the condition $V''(\phi_m) > 0$ means that the eigenvalues can lie close to 0 and s, but not r. We choose, without loss of generality, the sequence of eigenvalues to be decreasing (rather than increasing). Then there exists N such that

$$\phi_{-j}, \ldots, \phi_{-j+N-1} = s - \mathcal{O}\left(\mu^{-1}\right) \tag{6.27}$$

$$\phi_{-j+N}, \ldots, \phi_j = \mathcal{O}\left(\mu^{-1}\right) \tag{6.28}$$

that is the first N eigenvalues are close to s, and the remaining are close to 0. We can use (2.10) to compute the ϕ_m to first order in μ^{-1}. We see immediately that if $\phi_m = s - \mathcal{O}(\mu^{-p})$, then $\phi_{m+1} = s - \mathcal{O}(\mu^{-p+1})$ for $m \leq -j + N - 1$. Then we can write

$$\begin{aligned} -s + \mathcal{O}(\mu^{-1}) &= \phi_{-j+N} - \phi_{-j+N-1} = \frac{\mu}{\alpha_{-j+N-1}} \sum_{k=-j}^{-j+N-1} V'(\phi_k) \\ &= \frac{\mu}{\alpha_{-j+N-1}} V'(\phi_{-j+N-1}) + \mathcal{O}(\mu^{-1}) \\ &= \frac{\mu}{\alpha_{-j+N-1}} V''(s)(\phi_{-j+N-1} - s) + \mathcal{O}\left(\mu^{-1}\right) \end{aligned} \tag{6.29}$$

giving

$$\phi_{-j+N-1} = s - \frac{s\alpha_{-j+N-1}}{V''(s)} \mu^{-1} + \mathcal{O}\left(\mu^{-2}\right) \tag{6.30}$$

and

$$\phi_{-j+N-q} = s - \mathcal{O}(\mu^{-q}), \qquad q \geq 2. \tag{6.31}$$

Similarly,

$$\phi_{-j+N} = \frac{s\alpha_{-j+N-1}}{V''(0)} \mu^{-1} + \mathcal{O}\left(\mu^{-2}\right) \tag{6.32}$$

and

$$\phi_{-j+N-1+q} = \mathcal{O}\left(\mu^{-q}\right), \qquad q \geq 2. \tag{6.33}$$

We can use these results to evaluate the γ_m to first order, and find bounds on the C_m.

First, when $m + k \leq -j + N - 1$, we have $\gamma_m = \mu V''(s) + \mathcal{O}(1)$ giving

$$C_m = \mu V''(s) + \mathcal{O}(1), \qquad m \leq -j + N - k - 1. \tag{6.34}$$

Next, if $m = -j + N - k$, we find $\gamma_{-j+N-k} = -\alpha_{-j+N-1} + \mathcal{O}\left(\mu^{-1}\right)$, giving

$$C_{-j+N-k} = \alpha_{-j+N-k} + k(k-1) + \mathcal{O}\left(\mu^{-1}\right). \tag{6.35}$$

Then for $-j + N - k < m < -j + N - 1$, we have $\gamma_m = \mathcal{O}(\mu^{-1})$, and we can check inductively that

$$C_m > \alpha_m + k(k-1) + \mathcal{O}\left(\mu^{-1}\right), \qquad -j + N - k < m < -j + N - 1. \tag{6.36}$$

When $m = -j + N - 1$, we have $\gamma_{-j+N-1} = -\alpha_{-j+N-1} + \mathcal{O}(\mu^{-1})$ giving

$$C_{-j+N-1} > k(k-1) + \mathcal{O}\left(\mu^{-1}\right). \tag{6.37}$$

Finally, if $m > -j + N - 1$, we have $\gamma_m = \mu V''(0) + \mathcal{O}(1)$, and so

$$C_m = \mu V''(0) + \mathcal{O}(1)\,, \qquad m > -j + N - 1\,. \tag{6.38}$$

This completes the proof of theorem 3, since every C_n is positive at sufficiently large μ.

We have shown that a soliton which satisfies the assumptions of theorem 3 has positive definite stability functional on the rotation fixed space, and so is stable. In particular, the solitons which we constructed in theorem 1 are stable when μ is sufficiently large. These solitons cannot be deformed without increasing their energy, because this would require the stability functional to have zero eigenvalues after fixing the rotations.

7. Unstable multi-lump solutions

In this section we consider solutions of the field equations that describe separated lumps on the fuzzy sphere. As mentioned in the Introduction, a precise mathematical definition of a multi-soliton is lacking at finite non-commutativity but one can nevertheless look for solutions where the scalar field has nonvanishing values in well separated regions on the sphere. Perturbative calculations at large non-commutativity indicate that such lumps attract each other [12, 13] and therefore one would not expect to have solutions with multiple lumps at arbitrary locations. On symmetry grounds, however, one might anticipate that there could exist solutions with lumps in an unstable equilibrium. The simplest such solution would have two identical lumps, one at each pole of the sphere. Such a soliton would be rotationally invariant about the z-axis but also invariant under reflection about the equatorial plane. It would thus be a diagonal solution with the eigenvalues satisfying

$$\phi_m = \phi_{-m}\,. \tag{7.1}$$

If j is a half integer, the equations (2.10), (2.11) reduce consistently under this assumption to

$$\phi_{m+1} - \phi_m = \frac{R^2}{2\alpha_m} \sum_{i=-j}^{m} V'(\phi_i)\,, \qquad m = -j, \ldots, -\frac{3}{2} \tag{7.2}$$

with the constraint

$$\sum_{i=-j}^{-\frac{1}{2}} V'(\phi_i) = 0\,. \tag{7.3}$$

To see this, we use the property $\alpha_{-m} = \alpha_{m-1}$ to check that (2.10) also holds for $m \geq \frac{1}{2}$. When j is an integer, the equations become

$$\phi_{m+1} - \phi_m = \frac{R^2}{2\alpha_m} \sum_{i=-j}^{m} V'(\phi_i)\,, \qquad m = -j, \ldots, -1 \tag{7.4}$$

with the constraint

$$2\sum_{i=-j}^{-1} V'(\phi_i) + V'(\phi_0) = 0\,. \tag{7.5}$$

We can suitably adjust the set $A^{(j)}$ (equation (5.7)) to the altered difference equation and constraint. Then applying the argument of section 5 shows immediately that these soliton solutions do exist for sufficiently large θ. We see from the stability analysis in section 6 that such solitons are unstable since the eigenvalues do not form a monotonic sequence.

We can construct a bifurcation diagram for these solutions which must be a subset of the bifurcation diagram for the full problem. Then we see that they correspond precisely with certain trajectories in the full bifurcation diagram. In particular, the diagonal eigenstates of $[J_i,[J_i,\cdot]]$ corresponding to eigenvalue $k(k+1)$ with k even have the property $\phi_m = \phi_{-m}$, so every second bifurcation from $x = r$ corresponds to an unstable solution of this type.

By symmetry, one would expect to be able to go further, and exhibit unstable solitons with n lumps evenly spaced around a great circle. At fixed j, this n can be at most $2j$, as the fuzziness prevents resolution of more lumps. Here, we briefly report results for a specific example. We examine $j = 3/2$ and seek a 3-lump solution. We begin with a general diagonal matrix P, and a rotation through $2\pi/3$ about the x-axis, $\mathcal{R} = \exp(2\pi i J_x/3)$, and define a new matrix Q

$$Q = P + \mathcal{R}P\mathcal{R}^{-1} + \mathcal{R}^{-1}P\mathcal{R} \tag{7.6}$$

which is invariant under rotation through $2\pi/3$ about the x-axis. In this case, the situation is simplified because all such matrices Q, and kinetic terms $[J_i,[J_i,Q]]$ are simultaneously diagonalisable, and two of the eigenvalues are always identical. One can then solve the equations of motion for the eigenvalues, and find several solutions when μ is chosen large enough.

In the following section we take a limit of large R and j to blow up a small region close to the south pole into the Moyal plane. The multi-lump solutions constructed in this section will not survive this limit since any lump not sitting at the south pole will be moved to infinity. However, by symmetry one would expect the existence of an unstable solution consisting of an infinite chain of lumps finitely separated along a straight line in the Moyal plane, and its generalisation to a two-dimensional array of lumps.[2] Such solitons would of course have infinite energy.

8. The planar limit

An additional motivation for studying the fuzzy sphere is that, by blowing up a small region close to, say, the south pole, there is a weak convergence of the algebra to that of the Moyal plane [19, 34]. This means that the fuzzy sphere may be used to cut off the infinite dimensional algebra and provide an infra-red regularisation for field theory on the Moyal plane, but care is required when taking advantage of this limit because of the global differences between the sphere and the plane. In this section we show that the solitons constructed in section 5 do converge in a weak sense to solitons on the Moyal plane. The relevant limit is to send j and R to ∞, keeping the ratio

$$\theta = \frac{R^2}{2j} \tag{8.1}$$

[2]In [16] such periodic solutions are constructed in the limit of large θ.

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fixed. In theorem 1 of section 5, we found diagonal solitons for θ large enough. For each $0 < N \leq 2j$, we constructed a soliton $\psi_k^{(j)}$ with the first N eigenvalues lying in the interval (v, s) and with the others in $(0, u)$ (see figure 1). The constraint on θ is independent of j. We will show in theorem 4 below that the solutions $\psi_k^{(j)}$ have the property that there exist ψ_k with

$$\psi_k^{(j)} \to \psi_k \text{ as } j \to \infty \tag{8.2}$$

at fixed k. Assuming this result for the moment, we can take the limits of equation (5.2) at fixed q and discover

$$\psi_{q+1} - \psi_q = \frac{\theta}{2(q+1)} \sum_{k=0}^{q} V'(\psi_k)\,, \qquad q = 0, 1, \ldots \tag{8.3}$$

which is precisely the equation for a diagonal soliton on the Moyal plane with noncommutativity parameter θ [1, 14]. We will in addition see that $\psi_k \to 0$ as $k \to \infty$ which is necessary for the Moyal plane soliton to have finite energy.[3]

Theorem 4. *Send $j \to \infty$ and $R \to \infty$ keeping θ fixed and sufficiently large. In this limit, the matrix elements of the fuzzy sphere solitons constructed in section 5 converge to the matrix elements of a soliton on the Moyal plane.*

We begin by showing that $\psi_0^{(j)} = \inf A^{(j)}$ is a decreasing sequence and so has a limit. For each j, we define the interval $I_j = [x^{(j)}, y^{(j)}]$, where $x^{(j)}, y^{(j)}$ are the points defined in the proof of theorem 1, so that $A^{(j)} \subset I_j$. We recall that $x^{(j+1)} \leq x^{(j)}$ and $y^{(j+1)} \leq y^{(j)}$. Therefore, if $I_j \cap I_{j+1} = \emptyset$, we have $\inf A^{(j+1)} \leq \inf A^{(j)}$ and we are done.

If $I_j \cap I_{j+1} \neq \emptyset$, then pick $\alpha \in I_j \cap I_{j+1}$ and suppose $\alpha \notin A^{(j+1)}$. If we set $\psi_0^{(j)} = \psi_0^{(j+1)} = \alpha$, then $\alpha \notin A^{(j+1)}$ means that

$$\sum_{k=0}^{q} V'\left(\psi_k^{(j+1)}\right) \leq 0\,, \qquad \forall q\,. \tag{8.4}$$

We claim that

$$\psi_k^{(j)} \leq \psi_k^{(j+1)}\,, \quad \forall k. \tag{8.5}$$

By assumption, this is true for $k = 0$. We assume that (8.5) is true for $k \leq n$ and argue by induction. Since, for fixed k, $\psi_k^{(j+1)}$ and $\psi_k^{(j)}$ both lie in a domain in which V' is increasing, we have

$$V'\left(\psi_k^{(j)}\right) \leq V'\left(\psi_k^{(j+1)}\right) \tag{8.6}$$

for $k \leq n$. Then

$$\psi_{n+1}^{(j+1)} - \psi_n^{(j+1)} = \frac{\theta}{2(n+1)\left(1 - \frac{n}{2(j+1)}\right)} \sum_{k=0}^{n} V'\left(\psi_k^{(j+1)}\right)$$

[3] We note here that the stability analysis for the fuzzy sphere does not carry over to the Moyal plane by simply taking this limit. The condition given on R for stability does not match the condition on θ for the planar limit, and the more technically complicated analysis given in [15] is therefore still required for the Moyal plane.

$$\geq \frac{\theta}{2(n+1)(1-\frac{n}{2(j+1)})}\sum_{k=0}^{n} V'\left(\psi_k^{(j)}\right)$$

$$\geq \frac{\theta}{2(n+1)(1-\frac{n}{2j})}\sum_{k=0}^{n} V'\left(\psi_k^{(j)}\right) \tag{8.7}$$

and, by the induction hypothesis, this implies that

$$\psi_{n+1}^{(j+1)} \geq \psi_n^{(j)} + \frac{\theta}{2(n+1)(1-\frac{n}{2j})}\sum_{k=0}^{n} V'(\psi_k^{(j)})$$

$$= \psi_{n+1}^{(j)}, \tag{8.8}$$

establishing (8.5). Next, using (8.4) and (8.6), we have

$$\sum_{k=0}^{q} V'\left(\psi_k^{(j)}\right) \leq 0 \qquad \forall q\,. \tag{8.9}$$

which means $\alpha \notin A^{(j)}$.

So far, we have shown that if $\alpha \notin A^{(j+1)}$ then $\alpha \notin A^{(j)}$ and this implies

$$I_{j+1} \cap A^{(j)} \subset I_j \cap A^{(j+1)} \subset A^{(j+1)}. \tag{8.10}$$

Taking the infimum,

$$\inf\left(I_{j+1} \cap A^{(j)}\right) = \inf A^{(j)} \geq \inf A^{(j+1)}, \tag{8.11}$$

so $\psi_0^{(j)} = \inf A^{(j)}$ is a bounded, decreasing sequence, and so has a limit, $\psi_0^{(j)} \to \psi_0$, as $j \to \infty$.

The next stage in proving theorem 4 is to show that, for each fixed k, there is a ψ_k such that $\psi_k^{(j)} \to \psi_k$ as $j \to \infty$. This follows easily by induction. Assume it is true for $k \leq n$ and use (5.2) to note

$$\psi_{n+1}^{(j)} = \psi_n^{(j)} + \frac{\theta}{2(n+1)\left(1-\frac{n}{2j}\right)}\sum_{k=0}^{n} V'\left(\psi_k^{(j)}\right) \tag{8.12}$$

$$\to \psi_n + \frac{\theta}{2(n+1)}\sum_{k=0}^{n} V'(\psi_k) \tag{8.13}$$

as $j \to \infty$ by continuity of V'.

Finally, it remains to check the finite energy condition $\psi_k \to 0$ as $k \to \infty$. We first note that, since any soliton on the fuzzy sphere has its spectrum in the interval $[0, s]$, we have $\psi_k^{(j)} \geq 0$ for all k, j, and this implies $\psi_k \geq 0$ for all k. Similarly, since $\psi_{k+1}^{(j)} \leq \psi_k^{(j)}$, we also have $\psi_{k+1} \leq \psi_k$. Then ψ_{k+1} is a decreasing sequence which is bounded below and so there exists $\psi \geq 0$ with $\psi_k \to \psi$ as $k \to \infty$.

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By our original construction, we also have $\psi_k^{(j)} \leq u$ for all j and $k > N$, and so, taking the limit, $\psi_k \leq u$ for $k > N$. Then only $\psi_0, \ldots, \psi_N$ lie outside the region in which V' is increasing, so $V'(\psi_k) > V'(\psi)$ for $k \geq N + 1$. Substituting into (8.3) gives

$$\psi_{q+1} - \psi_q > \frac{\theta q}{2(q+1)} V'(\psi) + \frac{\theta}{2(q+1)} \sum_{k=0}^{N} V'(\psi_k) \tag{8.14}$$

for $q > N$, and taking the limit

$$0 \geq \frac{\theta}{2} V'(\psi) \qquad (\geq 0) \tag{8.15}$$

and we deduce that $\psi = 0$. So $\psi_k \to 0$ as $k \to \infty$, and this completes the proof of theorem 4.

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References

[1] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[2] G.H. Derrick, *Comments on nonlinear wave equations as models for elementary particles*, *J. Math. Phys.* **5** (1964) 1252.

[3] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043].

[4] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, *J. High Energy Phys.* **04** (2001) 001 [hep-th/0009142].

[5] D.J. Gross and N.A. Nekrasov, *Solitons in noncommutative gauge theory*, *J. High Energy Phys.* **03** (2001) 044 [hep-th/0010090].

[6] J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [hep-th/0010060].

[7] D.P. Jatkar, G. Mandal and S.R. Wadia, *Nielsen-olesen vortices in noncommutative abelian Higgs model*, *J. High Energy Phys.* **09** (2000) 018 [hep-th/0007078].

[8] D. Bak, *Exact multi-vortex solutions in noncommutative abelian- Higgs theory*, *Phys. Lett.* **B 495** (2000) 251 [hep-th/0008204].

[9] G.S. Lozano, E.F. Moreno and F.A. Schaposnik, *Nielsen-olesen vortices in noncommutative space*, *Phys. Lett.* **B 504** (2001) 117 [hep-th/0011205].

[10] R.J. Szabo, *Quantum field theory on noncommutative spaces*, hep-th/0109162.

[11] J.A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, hep-th/0102076.

[12] M. Spradlin and A. Volovich, *Noncommutative solitons on kaehler manifolds*, *J. High Energy Phys.* **03** (2002) 011 [hep-th/0106180].

[13] S. Vaidya, *Scalar multi-solitons on the fuzzy sphere*, *J. High Energy Phys.* **01** (2002) 011 [hep-th/0109102].

[14] B. Durhuus, T. Jonsson and R. Nest, *Noncommutative scalar solitons: existence and nonexistence*, *Phys. Lett.* **B 500** (2001) 320 [hep-th/0011139].

[15] B. Durhuus, T. Jonsson and R. Nest, *The existence and stability of noncommutative scalar solitons*, hep-th/0107121.

[16] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons*, hep-th/0103256.

[17] U. Lindstrom, M. Roček and R. von Unge, *Non-commutative soliton scattering*, *J. High Energy Phys.* **12** (2000) 004 [hep-th/0008108].

[18] B. Durhuus and T. Jonsson, *A note on noncommutative scalar multisolitons*, *Phys. Lett.* **B 539** (2002) 277 [hep-th/0204096].

[19] C.S. Chu, J. Madore and H. Steinacker, *Scaling limits of the fuzzy sphere at one loop*, *J. High Energy Phys.* **08** (2001) 038 [hep-th/0106205].

[20] T. Araki and K. Ito, *Scattering of noncommutative (N,1) solitons*, *Phys. Lett.* **B 516** (2001) 123 [hep-th/0105012].

[21] L. Hadasz, U. Lindstrom, M. Roček and R. von Unge, *Noncommutative multisolitons: moduli spaces, quantization, finite theta effects and stability*, *J. High Energy Phys.* **06** (2001) 040 [hep-th/0104017].

[22] A. Solovyov, *On noncommutative solitons*, *Mod. Phys. Lett.* **A 15** (2000) 2205 [hep-th/0008199].

[23] Y. Matsuo, *Topological charges of noncommutative soliton*, *Phys. Lett.* **B 499** (2001) 223 [hep-th/0009002].

[24] M.G. Jackson, *The stability of noncommutative scalar solitons*, *J. High Energy Phys.* **09** (2001) 004 [hep-th/0103217].

[25] C.G. Zhou, *Noncommutative scalar solitons at finite theta*, hep-th/0007255.

[26] Y. Hikida, M. Nozaki and T. Takayanagi, *Tachyon condensation on fuzzy sphere and noncommutative solitons*, *Nucl. Phys.* **B 595** (2001) 319 [hep-th/0008023].

[27] K. Hashimoto and K. Krasnov, *D-brane solutions in non-commutative gauge theory on fuzzy sphere*, *Phys. Rev.* **D 64** (2001) 046007 [hep-th/0101145].

[28] H. Steinacker, *Aspects of the q-deformed fuzzy sphere*, *Mod. Phys. Lett.* **A 16** (2001) 361 [hep-th/0102074].

[29] Y. Kimura, *Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model*, *Prog. Theor. Phys.* **106** (2001) 445 [hep-th/0103192].

[30] C.T. Chan, C.M. Chen, F.L. Lin and H.S. Yang, *CP^n model on fuzzy sphere*, *Nucl. Phys.* **B 625** (2002) 327 [hep-th/0105087].

[31] B.P. Dolan, D. O'Connor and P. Prešnajder, *Matrix ϕ^4 models on the fuzzy sphere and their continuum limits*, *J. High Energy Phys.* **03** (2002) 013 [hep-th/0109084].

[32] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, *Noncommutative gauge theory on fuzzy sphere from matrix model*, *Nucl. Phys.* **B 604** (2001) 121 [hep-th/0101102].

[33] B.P. Dolan, D. O'Connor and P. Prešnajder, *Matrix models on the fuzzy sphere*, hep-th/0204219.

[34] J. Madore, *The fuzzy sphere*, *Class. and Quant. Grav.* **9** (1992) 69.

[35] H. Grosse, C. Klimčík and P. Prešnajder, *Towards finite quantum field theory in noncommutative geometry*, *Int. J. Theor. Phys.* **35** (1996) 231 [hep-th/9505175].

[36] A. Perelomov, *Generalized coherent states and their applications*, Springer-Verlag, 1986

[37] F.A.Berezin *Sov. Math. Izv.* **38** (1974) 1116.